Math 340 Summer '18
Midterm
2018-07-20

KEY

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

- There are 5 problems on this exam. Be sure you have all 5 problems on your exam.
- All vector spaces are defined over a field F which you can take to be $\mathbf{R}$ or $\mathbf{C}$.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

1. (10 points) Determine whether the following is True or False. You do not need to justify your answer. Write T for True and F for False.

   (a) **F** If S is a spanning subset of V then any subset of S also spans V.

   (b) **T** A linear transformation $T : V \to W$ is one-to-one if and only if the nullity of T is 0.

   (c) **F** If $T : V \to W$ is a linear transformation between infinite dimensional spaces, then $N(T)$ must also be infinite dimensional.

   (d) **F** A linear transformation $T : V \to W$ is one-to-one if and only if $T(0) = 0$.

   (e) **T** Let $T : P_2(\mathbf{R}) \to \mathbf{R}^2$ be a linear map with the property $T(1) = (1, 2)$, $T(x) = (-1, 1)$, and $T(x^2) = (0, 1)$. Then $T(2x^2 + 1) = (1, 4)$.

2. (10 points) Let $T : P_2(\mathbf{R}) \to \mathbf{R}^2$ be a linear transformation (you may assume this) given by

$$T(p) = (p(1), p'(1)).$$

So the first component of $T(p)$ is p evaluated at 1 and the second component is the derivative of p evaluated at 1. Let $\alpha = \{1, x, x^2\}$ be an ordered basis for $P_2(\mathbf{R})$ and $\beta = \{(1,0), (0,1/2)\}$ be an ordered basis for $\mathbf{R}^2$.

(a) What is $[T]_\alpha^\beta$?

Solution: We first compute $T(1) = (1,0)$, $T(x) = (1,1)$, and $T(x^2) = (1,2)$. Since $(a,b) = a(1,0) + 2b(0,1/2)$, we know that $[(a,b)]^\beta = (a, 2b)$. Putting all this together,

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

(b) Prove that T is onto without using a pivot argument.

Solution: Using the last part, $T(1) = (1,0)$ and $T(x) = (1,1)$. Since $(1,0)$ and $(1,1)$ spans the codomain $\mathbf{R}^2$, the range of T contains a spanning set so T is onto.

(c) What is the nullity of T?

Solution: By the rank-nullity theorem, $\dim P_2 = \text{rank}(T) + \text{nullity}(T)$. We know that $\dim P_2 = 3$. From the last part, $\text{rank}(T) = 2$. Therefore, $\text{nullity}(T) = 1$.

3. (10 points) Let $T : V \to W$ be a linear transformation between vector spaces V and W. Let W_1 be a subspace of W. Prove that $S = \{v \in V : T(v) \in W_1\}$ is a subspace of V.

Solution: We proceed by checking the 3 properties:

- Since $T(0) = 0 \in W_1$, we know that $0 \in S$.
- Suppose $x, y \in S$. Since T is linear, $T(x + y) = T(x) + T(y)$. Since $x, y \in S$, we know that $T(x) \in W_1$ and $T(y) \in W_1$. Because W_1 is a subspace, $T(x) + T(y) \in W_1$ so $T(x + y) = T(x) + T(y) \in W_1$. This means $x + y \in S$.
- Suppose $x \in S$ and $c \in F$. Since T is linear, $T(cx) = cT(x)$. Since $x \in S$, we know $T(x) \in W_1$. Because W_1 is a subspace, $cT(x) \in W_1$ so $T(cx) = cT(x) \in W_1$. This means $cx \in S$.

4. (10 points) Let $T : V \to W$ be a linear transformation between vector spaces V and W. Prove that T is onto if and only, $T(S)$ spans W for any spanning subset S of V.

Solution: First suppose T is onto. Let S be any spanning subset of V. We wish to show $T(S)$ spans W. Let $w \in W$. Since T is onto, there exists a v such that $T(v) = w$. Since S is spanning, there exists $a_1, \ldots, a_n \in F$ and $s_1, \ldots, s_n \in S$ such that

$$v = a_1 s_1 + \cdots + a_n s_n.$$

By applying T and both sides and using linearity of T,

$$w = T(v) = a_1 T(s_1) + \cdots + a_n T(s_n).$$

Hence, $w \in \operatorname{span} T(S)$. Therefore, $T(S)$ spans W.

Conversely, suppose $T(S)$ spans W for any spanning subset S of V. In particular, this means that $T(V)$ is spanning. So $\operatorname{span} T(V) = W$. But $T(V)$ is subspace so it is equal to its span. Hence, $T(V) = \operatorname{span} T(V) = W$, which implies T is onto.

5. (10 points) Let X, Y, Z be finite dimensional vector spaces. Let $T : X \to Y$ and $S : Y \to Z$ be linear transformations so that

   - T is one-to-one,
   - S is onto,
   - $R(T) = N(S)$.

   Prove that $\dim Y = \dim X + \dim Z$.

Solution: We see 2 linear transformations. We see information about their rank and nullity. We see the word dimension 3 times in an equality. We strongly suspect the usage of the rank-nullity theorem.

By the rank-nullity theorem applied to S, $\dim Y = \operatorname{rank} S + \operatorname{nullity} S$. Since S is onto, $\operatorname{rank} S = \dim Z$. Because $N(S) = R(T)$, $\operatorname{nullity} S = \operatorname{rank} T$. Since T is one-to-one, $\operatorname{nullity} T = 0$, so rank-nullity implies $\dim X = \operatorname{rank} T$. Putting all of this together, we deduce that $\dim Y = \dim X + \dim Z$.