Name: Key Signature:______________________

1. (30 points) In the following, each correct answer is worth 2 points. There is no penalty for incorrect answers. You do not need to justify your answers.

   (a) The rank of a matrix is

   - ☐ the dimension of its null space.
   - ✓☒ the dimension of its range.
   - ☐ both of the above
   - ☐ neither of the above

   (b) Write down an *orthogonal* basis for span$\left\{ \begin{pmatrix} 7\\0\\1 \end{pmatrix}, \begin{pmatrix} 5\\0\\2 \end{pmatrix}, \begin{pmatrix} 2\\0\\-3 \end{pmatrix}, \begin{pmatrix} 5\\0\\4 \end{pmatrix}, \begin{pmatrix} -7\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0 \end{pmatrix} \right\}$.

   $\left\{ \begin{pmatrix} 1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1 \end{pmatrix} \right\}$ Version 2 : $\left\{ \begin{pmatrix} 1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0 \end{pmatrix} \right\}$

   (c) Let $A$ be a $3 \times 3$ matrix whose only eigenvalue is 5, with associated eigenspace all of $\mathbb{R}^3$. Find $A$.

   $$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

   (d) Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix}$ be a square matrix with columns $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, where $\mathbf{a}_1 = \mathbf{a}_2 + \mathbf{a}_3$. Find $\det(A)$.

   0 ($A$ has linearly dependent columns $\Rightarrow A$ is noninvertible)

   (e) Let $A$ be a $4 \times 4$ matrix with $\text{rank}(A) = 4$. What is $\text{rank}(A^{-1})$?

   4 ($A^{-1}$ is invertible, $4 \times 4 \Rightarrow \text{Rank}(A^{-1}) = 4$)

   Version 2 : 3

   (f) If $A = \begin{pmatrix} 2 & 0 & 5 & 1 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 8 \end{pmatrix}$, list all the eigenvalues of $A$.

   2, 3, 5, 8 ($A$ is upper triangular) Version 2 : 2, 3, 4, 7

   (g) Give an example of a matrix whose domain is $\mathbb{R}^3$ and range is span$\left\{ \begin{pmatrix} 1\\0 \end{pmatrix} \right\}$.

   $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ Version 2 : $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$

(h) Find a vector $\mathbf{v}$ so that $\left\{\begin{pmatrix}1\\-1\end{pmatrix}, \mathbf{v}\right\}$ is an *orthogonal* basis for $\mathbb{R}^2$.

$\vec{v} = \begin{pmatrix}1\\1\end{pmatrix}$

(i) Give an example of a nonzero vector $\mathbf{v}$ that lies in $S^\perp$, if $S = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

$\vec{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

(j) If $S = \text{span}\left\{\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}17\\0\\-8\end{pmatrix}\right\}$ $= \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$, find $\text{proj}_S \begin{pmatrix}1\\3\\-1\end{pmatrix}$.

$\begin{pmatrix}1\\0\\-1\end{pmatrix}$

Version 2 : $\begin{pmatrix}0\\3\\-1\end{pmatrix}$

(k) Write down a basis for the null space of $\begin{pmatrix}2 & 0 & 0 & 0 & 0\\-1 & 0 & 0 & 1 & 0\end{pmatrix}$.

$\left\{\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\} = \{\vec{e}_2, \vec{e}_3, \vec{e}_5\}$

Version 2: $\{\vec{e}_2, \vec{e}_4, \vec{e}_5\}$

(l) If $A = \begin{pmatrix}1/3 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1/2\end{pmatrix}$, what is $A^{-1}$?

$\begin{pmatrix}3 & 0 & 0\\0 & 1 & 0\\0 & 0 & -2\end{pmatrix}$

Version 2 : $\begin{pmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & -3\end{pmatrix}$

(m) Let $A = \begin{pmatrix}3 & 0\\0 & 1/3\end{pmatrix}$. For which vectors $\mathbf{x}$ does $\lim_{k\to\infty} A^k\mathbf{x}$ exist?

$\vec{x} \in \text{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$

(n) Let $A$ be a $2 \times 2$ matrix with eigenvalues 0 and 5. What is the rank of $A$?

1 (0 is an e-value $\Rightarrow$ Rank not 2
Not the zero matrix $\Rightarrow$ Rank not 0)

(o) If $A$ is a noninvertible square matrix, then the system $A\mathbf{x} = \mathbf{0}$ has

- ☐ no solution.
- ☐ a unique solution.
- ☑ infinitely many solutions.

(At least one solution — the trivial one — exists. In echelon form, there is at least one zero r

2. (7 points) Let $A = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix}$. Find all eigenvalues of $A$ and their associated eigenspaces.

$$(4-\lambda)(-1-\lambda) + 6 = 0$$

$$\Leftrightarrow \lambda^2 - 3\lambda + 2 = 0$$

$$\Leftrightarrow \lambda = 1, 2$$

$\underline{\lambda = 1}$: $\begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x_2 = \frac{-3x_1}{2}$

$\rightsquigarrow$ eigenspace $= \text{span}\left\{\begin{pmatrix} 1 \\ -3/2 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right\}$

$\underline{\lambda = 2}$: $\begin{pmatrix} 2 & 2 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Leftrightarrow x_2 = -x_1$

$\rightsquigarrow$ eigenspace $= \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

3. (3 points) Compute $\det(A)$, if $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ -1 & 5 & 4 \end{pmatrix}$.

$$1(8-5) - 3(0+1) = 0$$

4. (5 points) Let $S = \text{span}\left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\}$. Compute $\text{proj}_S \begin{pmatrix}0\\0\\1\end{pmatrix}$.

$$\underbrace{\left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\}}_{\text{orthogonal basis}}$$

$$\frac{1}{\cancel{\sqrt{\ }}6}\begin{pmatrix}1\\2\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1/2\\0\\1/2\end{pmatrix}$$

Version 2 :

$$\frac{2}{6}\begin{pmatrix}1\\2\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

5. (5 points) If $A$ is a matrix such that $A\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ and $A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, find $A$.

$$A = \begin{pmatrix}a & 0\\b & 1\end{pmatrix}$$

$$\begin{pmatrix}a & 0\\b & 1\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Longrightarrow \begin{matrix}a = 0\\b = 2\end{matrix}$$

$$\Longrightarrow A = \begin{pmatrix}0 & 0\\2 & 1\end{pmatrix}$$

Version 2 : $\cancel{\begin{pmatrix}2 & 1\\0 & 0\end{pmatrix}}$

$$\begin{pmatrix}1 & 2\\0 & 0\end{pmatrix}$$

6. (10 points) Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) (5 points) Find a basis for Range($A$).

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

linearly independent

A basis is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(b) (5 points) Find a basis for Null($A$).

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad : \quad \begin{aligned} x_3 &= s \\ x_2 &= x_3 = s \\ x_1 &= -x_2 = -s \end{aligned}$$

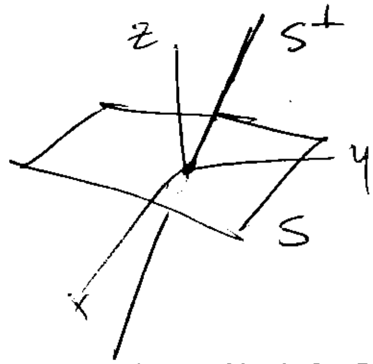
$$\Rightarrow \text{Null}(A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

7. (10 points) Let $S = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right\}$.

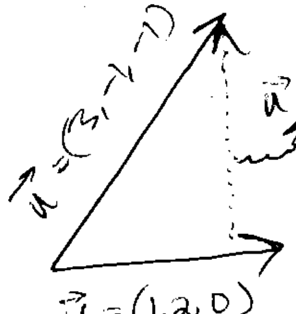
(a) (4 points) What is $\dim(S^{\perp})$? Explain either in 1-2 sentences or by drawing a picture.

$S = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right\} \Rightarrow \dim S = 2$

$\Rightarrow \dim S^{\perp} = 3 - 2 = 1$




(b) (6 points) Find an *orthogonal* basis for $S$. It may help to recall that $(\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}) \cdot \mathbf{v} = 0$, for any nonzero vectors $\mathbf{u}$ and $\mathbf{v}$.




$\vec{u} - \text{proj}_{\vec{v}}\vec{u} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} - \frac{3-2}{5}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1/5 \\ 2/5 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 14/5 \\ -7/5 \\ -1 \end{pmatrix}$

An orthogonal basis is

$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 14/5 \\ -7/5 \\ -1 \end{pmatrix} \right\}$

8. (10 points) Find a matrix $A$ such that

- $\text{Null}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$,
- $\text{Range}(A) = (\text{Null}(A))^{\perp}$, and
- 12 is an eigenvalue of $A$.

- Find $\text{Null}(A)^{\perp}$ : Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \text{Null}(A)^{\perp}$

$$\Rightarrow a - c = 0 \Rightarrow a = c$$
$$\& \quad b - c = 0 \Rightarrow b = c$$

$$\Rightarrow \text{Range}(A) = \text{Null}(A)^{\perp} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

- $\Rightarrow A = \begin{pmatrix} d & e & f \\ d & e & f \\ d & e & f \end{pmatrix}$.

- $A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow d = f$

$A \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow e = f$

$\Rightarrow A = \begin{pmatrix} d & d & d \\ d & d & d \\ d & d & d \end{pmatrix}$

e-vector

- $\begin{pmatrix} d & d & d \\ d & d & d \\ d & d & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 12 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow 3d = 12$
$\Rightarrow d = 4$

e-value

$\Rightarrow A = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix}$.

Version 2 : $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$