Math 308G - Winter 2018
Final
2018-03-13

KEY

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
Total:	72	

- There are 5 problems on this exam. Be sure you have all 5 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0.
- When I define a variable, it is defined for that whole question. The A defined in Question 2 is the same for each part.
- I treat row and column vectors as the same.
- For any linear transformation T, there exists a matrix A such that $T(x) = Ax$. I defined the determinant, rank, and nullity of T using A. This means,

$$\det(T) = \det(A), \quad \text{rank}(T) = \text{rank}(A), \quad \text{nullity}(T) = \text{nullity}(A).$$

1. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE". You do not need to justify your answers.

(a) (2 points) If possible, give an example of a 2×2 matrix A that is not diagonalizable but A^2 is diagonalizable.

Solution: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = 0$ which is diagonalizable.

(b) (2 points) If possible, give an example of a 2×2 matrix A such that $A^2 = I_2$ and $\text{nullity}(A) = 1$.

Solution: NOT POSSIBLE. If $A^2 = A$, then A is invertible so the nullity is 0.

(c) (2 points) If possible, give an example of a 2×2 matrix with distinct eigenvalues that is not invertible.

Solution: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(d) (2 points) If possible, give an example of linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\text{rank}(T) = \text{rank}(S) = \text{rank}(T \circ S) = 1$.

Solution: $T(x, y) = S(x, y) = (x, 0)$.

(e) (2 points) If possible, give an example a 2×4 matrix A such that $\text{rank}(A) = \text{nullity}(A)$.

Solution: Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

(f) (2 points) If possible, give an example of a 2×2 matrix A such that 1 is not an eigenvalue of A^2 but 1 is an eigenvalue of A^4. (Think geometrically).

Solution: Let A be rotation by $\pi/2$.

2. Perform the following computations.

(a) (6 points) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}.$$

- What is the reduced echelon form of A?

Solution:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Does $Ax = (1, 2, 1)$ have a solution? If so, give the general solution.

Solution: Yes.

$$(1, 1, 0) + s_1(-2, -1, 1)$$

- Does $Ax = (1, 1, 1)$ have a solution? If so, give the general solution.

Solution: No.

- What is a basis for row(A)?

Solution:

$$\{(1, 0, 2), (0, 1, 1)\}$$

(b) (6 points) Let

$$B = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- What is B^{-1}?

Solution:

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

- What is the charateristic polynomial of B^2? (Hint: Be careful about the sign.)

Solution: The eigenvalues of B^2 are the square of the eigenvalues of B. The charateristic polynomial is

$$(4 - \lambda)(1 - \lambda)^2$$

3. Let A and B be equivalent matrices given by

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 1 & 1 & 3 \\ 2 & 5 & 6 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

Denote the columns of A by a_1, a_2, a_3, a_4.

(a) (3 points) Give a basis for null(A).

Solution:

$$\{(-3, -8, 5, 1)\}$$

(b) (3 points) Give the general solution to $Ax = a_2$.

Solution:

$$(0, 1, 0, 0) + s_1(-3, -8, 5, 1)$$

(c) (3 points) Give the general solution to $2Ax - a_3 = a_1 + a_2$.

Solution:

$$(1/2, 1/2, 1/2, 0) + s_1(-3, -8, 5, 1)$$

(d) (3 points) It turns out $e_1 = (1, 0, 0, 0) \notin \text{col}(A)$. Give a vector v such that $v \neq e_1$ and $Ax = e_1 - v$ has a solution.

Solution: Take e_1 with any vector in col(A). For example,

$$v = e_1 + a_1$$

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformations given by

$$T(x, y, z) = (x + 2z, y + z, x + y + z, z)$$

and

$$S(x, y, z) = (2x + y + z, y + z, y + z)$$

(a) (2 points) There exists a matrix A such that $T(v) = Av$. What is A?

Solution:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) (2 points) There exists a matrix B such that $S(v) = Bv$. What is B?

Solution:

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(c) (4 points) These should be quick questions.

- What is rank(S)?

Solution: 2

- What is det(S)?

Solution: 0

- What is nullity($2S$)?

Solution: 1

- What are 2 different eigenvalues of S?

Solution: $0, 2$

(d) (4 points) Recall that $(T \circ S)(v) = T(S(v))$.

- What is the rank of $T \circ S$?

Solution: 2. See next part for explanation.

- What is a basis for range($T \circ S$)? (Hint: Look at the 2nd two columns of B).

Solution: The range of $T \circ S$ is spanned by $\{(T \circ S)(e_1), (T \circ S)(e_2), (T \circ S)(e_3))\}$ but $S(e_2) = S(e_3)$. So the range of $T \circ S$ is spanned by $\{(T \circ S)(e_1), (T \circ S)(e_2)\}$.
Then choose your favorite method to determine that these are linearly independent.

5. Let

$$A = \begin{bmatrix} 0 & -1 & 3 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

be a matrix which decomposes as $A = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Let u_1, u_2, u_3, u_4 be the columns of U and $\mathcal{B} = \{u_1, u_2, u_3, u_4\}$.

(a) (6 points) Fill out the following table.

Eigenvalue λ	Alg. Multiplicity of λ	Geo. Multiplicity of λ	A Basis for E_λ
0	1	1	$\{u_1\}$
-1	1	1	$\{u_2\}$
2	2	2	$\{u_3, u_4\}$

(b) (3 points) What is a basis for range(A)?

Solution: $\{(0,1,1,1), (0,0,1,1), (0,0,0,1)\}$.

(c) (3 points) Let $x = u_1 + u_2 + u_3$. What is $[A^{18}x]_{\mathcal{B}}$? You are allowed to have exponents in your answer.

Solution: $(0, 1, 2^{18}, 0)$.

6. Let A, u_1, u_2, u_3, u_4 be as defined in Question 5.

   (a) (2 points) What is $\det(A)$?

   Solution: 0.

   (b) (2 points) What is $\det(A + 2I)$?

   Solution: The eigenvalues of $A + 2I$ are $2, 1, 4, 4$. The determinant is then $2 \cdot 1 \cdot 4 \cdot 4$.

   (c) (2 points) What is $\text{rank}(A)$?

   Solution: This is just the rank of D. Or the number of nonzero eigenvalues of A, counting multiplicities.

   (d) (2 points) What is $\text{rank}(A - 2I)$?

   Solution: This is just the rank of $[A - 2I]_B$. Or the number of nonzero eigenvalues of $A - 2I$, counting multiplicities.

   (e) (2 points) Does $Ax = -u_2 + u_3 - 4u_4$ have a solution? If so, express it as a linear combination of u_1, u_2, u_3, u_4.

   Solution: $x = u_2 + \frac{1}{2}u_3 - 2u_4$.

   (f) (2 points) Does $Ax = 2u_1 + u_2 + u_3$ have a solution? If so, express it as a linear combination of u_1, u_2, u_3, u_4.

   Solution: No. The column space of A does not contain any vector with a nonzero first component.